

Secondary OTS Masterclass: The Maths of Voting

Session Leader Notes

Thank you very much for leading this Masterclass. We hope that you enjoy working with this material as much as we enjoyed putting the activities together. We do appreciate all the effort that our volunteers put into bringing inspirational Mathematics Masterclasses to students around the country. Don't forget that we'd love to know your thoughts on the Masterclass – more detail at the end of this section.

Inspiration:

Voting is a key part of democracy, the system that allows all eligible citizens to decide upon who will run the country. For example, general elections within the UK are generally held every five years and allow citizens to vote for who will represent their constituency as a Member of Parliament, and as such which party will have majority in the House of Commons.

This session introduces students to electoral maths, by exploring how different voting systems, including First Past the Post, Alternative Vote, and Borda, calculate who wins an election. Students will evaluate the advantages and disadvantages of these different electoral systems as well as evaluate to what extent these systems are fair.

Students will then use the knowledge and skills from this exploration to discuss and develop their own voting method.

Length: 2 and a half hours

Overview of Activities:

- **Can you gerrymander?** Students learn what gerrymandering means - changing the boundaries of a constituency to help win an election, by removing or adding voters that tend to vote a certain way. Students work through a mathematical/logic puzzle (Worksheet 1) to demonstrate how gerrymandering works in practice.
- **Alternative Vote (AV).** Students learn how First Past the Post, in which the candidate with the highest number of votes wins, works, as well as its flaws. This is then contrasted with Alternative Vote, in which the candidate with the majority of votes, after an order of preferences is given, wins. Students work through an example and then use Alternative Vote to work out the winner in a vote of favourite pizza toppings (Worksheet 2).
- **Borda.** Students learn what the Borda system/Borda count is; where candidates are awarded points based on voters' preferences. Students use Borda to determine the winner in the pizza toppings vote (Worksheet 3), and contrast this to the winner when First Past the Post and Alternative Vote are used.
- **Deciding on a system.** In groups, students run a mock election using their choice of the systems they have learned about, or a randomly selected other system (Worksheet 4, Instructions for Random Systems sheet, Ballot Handout).

General Masterclass resources needed:

- Register of children
- Consent forms and emergency information to hand
- Stickers and markers for name badges
- Adult register
- Ri child protection policy
- Paper and pencils/whiteboards for workings
- 2 different coloured post-it note pads
- Settling activity if not included in specific Masterclass
- Drinks and biscuits

Specific resources needed (enough for the number of students attending):

- 1 copy of Worksheets 1, 2 and 3 per student;
 - Please print **Worksheet 1 in colour** where possible
- 1 copy of Worksheet 4 per group of 4 students (1 side);
- 1 copy of Random systems instructions per group of 4 students;
- 3 copies of Ballot handout per student (as each student will need to vote 3 times);
- Around 6 six-sided dice on hand to give to students if they need them for the last activity
 - If dice are unavailable, use an equivalent for producing 50/50 odds e.g.,
 - coins (heads or tails)
 - a deck of playing cards (draw a red or black suit, for example)
- Students will need a pen/pencil;
- Students will also need a calculator.

Support resources:

- PowerPoint slides
- Helper notes
- Session script
- Additional information

Things to prepare in advance:

- Print worksheets and gather the complete list of resources as detailed above
- Print helper notes, session script, and other resources
- Gather general Masterclass resources

Ask the Ri

Don't forget to collect any questions which arise, and email them to the Masterclass team at the Royal Institution: masterclasses@ri.ac.uk

Feedback

We would very much welcome your feedback on this session. If you have time, please collect feedback from the students at the end of the Masterclass and send it through to us. We would also appreciate feedback on how you have used the session, what you think worked well and what improvements would be useful.

Time plan of Masterclass:

Slides & Time	Overview	Activity (see script for further details)
Slide 1 5 minutes (5)	The Maths of Voting Introduction	Today's objective: explore the maths behind voting systems and evaluate their fairness. Ask students to discuss their favourite colour (you can adapt this and make it any trivial question like favourite crisps or TV show). Get them to give their suggestions for a bit, and then tell them that by the end of the session they will understand how best to determine a winner.
Slides 2-3 5 minutes Slide 2 2 minutes	Voting examples	Show the voting examples on the slides; from top-left clockwise: - Polling station – UK General Election - I'm A Celebrity, Get Me Out of Here! - a UK game show in which viewers vote for their favourite contestants to stay in the competition, or to participate in trials. - Show of hands – vote for most popular thing in a group, may be something students have done in class before. Pictured is the European Parliament using show of hands - Poll card for UK Brexit Referendum
Slide 3 3 minutes (10)	Why do we vote?	Voting is important – in a democratic society, citizens must be allowed to have their opinions taken into consideration on important topics. Maths and voting are connected, in that maths can be used to design a voting system and judge its fairness.
Slide 4 5 minutes (15)	Are elections fair?	Sometimes voting systems result in unexpected winners. In 2016, Trump won the US Presidential Election because he won the most votes in enough states, even if he did not win the most votes overall. <i>If you feel able to explain the Electoral College, feel free to do so – there is more information in the Additional Information document, however you do not need to.</i> The photo in the middle is Theresa May, the Conservative leader and Prime Minister at the time. The 2017 general election resulted in a hung Parliament, where no party holds a majority of seats, making decision-making difficult. In this case 326 seats were needed to win, but no party won that many seats, so this led to a "coalition" between the Conservatives (318 seats) and the DUP (10 seats), to give them 328 seats in total. On the right is Alexander Lukashenko, the first and only President of Belarus since 1994. In 2020 when Lukashenko won by over 80%, many countries spoke out accusing him of election fraud. Typically, a victory with over 60% of the popular vote is considered enough to raise eyebrows. Even where the numbers are in one candidate's favour, there may be more to it behind the scenes. Summarise that electoral systems can create unexpected/atypical results.

Slides & Time	Overview	Activity (see script for further details)
Slide 5 5 minutes (20)	What makes a voting system fair?	Ask students to discuss with their neighbours what characteristics make a system fair, and then ask them to share their ideas with the rest of the group.
Slide 6 5 minutes (25)	According to Kenneth Arrow	Introduce Arrow's theorem for evaluating whether a system is fair – 1. All votes must be equal (non-dictator) 2. If everyone prefers one candidate, then that candidate must win (unanimity) 3. There should be one clear winner (universality) 4. The ranking of A and B should only depend on how individual votes ranked them (independence of irrelevant alternatives) Is there anything not listed here that you think should be? Things like anonymity are not considered here.
Slides 7-8 5 minutes Slide 7 3 minutes	Arrow's theorem and ice cream	Walk students through a practical application of Arrow's theorem, using ice cream as an example on the slide: - The three siblings Kendall, Siobhan and Connor have ranked flavours of ice cream from favourite on the left to the least favourite on the right - Everyone's votes must be equal, and if there is a clear favourite, then that one – chocolate – should win. According to the last part of Arrow's Theorem, if after chocolate had won another flavour was introduced (e.g., lemon), only chocolate or lemon could be the winners there.
Slide 8 2 minutes (30)	Voting Systems	Introduce the names of the three systems to be examined, and briefly some examples of their current uses. We will be looking at how these systems work and using them to solve problems where people have different preferences. Tell students that there are many more voting systems beyond these. <i>More information on these systems is available on the Additional Information sheet.</i>
Slides 9-13 10 minutes Slide 9 3 minutes	First Past the Post (FPTP)	Introduce students to First Past the Post (FPTP), in which the voters mark a candidate as their favourite option, and the option with the most first-preference votes wins. <i>More information on this is available on the Additional Information sheet.</i> If you would like, you can run a mock election in class – allow students to put their hand up for their favourite of the pizzas along the side. Whichever has the most votes wins.

Slides & Time	Overview	Activity (see script for further details)
Slide 10 - 12 4 minutes	What pizza shall we order?/Ham & pineapple wins!	Introduce students to the example of a family of 12 trying to decide what toppings of pizza to order. On the slide, students can see each person's choice. Get students to work out which pizza wins using the First Past the Post method (ham & pineapple wins, with the most votes – 3 votes). Click the slide to show the 3 votes, and then again to show the winning pizza. If you ran a mock election earlier also, ask students to think about whether that outcome was fair, and how they felt about the results of it.
Slide 13 3 minutes (40)	Most people voted against it	Given that ham & pineapple only received 3 out of 12 votes (or 25% of the vote) and that most people voted against it, ask students if they think this is a fair outcome. Ask the students what happens if someone can't eat ham or is allergic to pineapple? Is this a satisfactory outcome – why/why not?
Slide 14-15 5 minutes (45)	First Past the Post at national level	Explain how First Past the Post works at the national level, by describing how each constituency, a group of voters in a specific area, elects one person, a Member of Parliament (MP), to represent their area in parliament. Explain that in UK elections, there are effectively over 600 mini-elections happening across the country where voters can choose between different candidates for their constituency. They typically represent different political parties, such as Labour, Conservatives, and Liberal Democrats. Whoever wins the most votes in a constituency become the MP there, and that is classed as one seat in parliament for their party. This is a literal seat in the room, called the House of Commons. For a party to form a government, they need to have over half of the possible seats (in the case of the 2024 election, they needed a minimum of 326 seats). <i>Slide 15:</i> Analyse the graph with the students and ask them to think about the difference between percentage of votes and seats for the top five most voted parties in the last election. You could highlight how the difference in vote share between Reform and the Liberal Democrat party is minimal, in comparison to their big difference in seats share. Explain that it is because the votes for Reform were spread across the country in different constituencies, whereas votes for liberal democrats were typically within constituencies with a large liberal democrat voting population.
Slide 16 5 minutes (50)	Gerrymandering	Explain what gerrymandering is – changing the boundaries of a constituency to help win an election, by removing or adding voters that tend to vote a certain way. Follow the link on the slide to The Guardian's article on a real-life case of gerrymandering in North Carolina to demonstrate how gerrymandering can turn a predominantly Democrat area into a Republican one – and the impact of this for election results. <i>The Guardian article is very visual/interactive so no need to explain much yourself.</i>

Slides & Time	Overview	Activity (see script for further details)
Slide 17-18 10 minutes (60)	Can you gerrymander? /Solutions Worksheet 1	Hand out Worksheet 1: Can you gerrymander? You should ask students: - If there are six constituencies, how many does Magenta need to have a majority overall? (4) - What happens if there is a tie? <i>Slide 21</i> Once everyone is finished, ask students if everyone managed to gerrymander so that Magenta has a majority of constituencies in Maps 1 and 2. You may wish to take some example solutions from the students. Ask students if they managed to do the same for Map 3 (they cannot, it is mathematically impossible to do so). You may wish to ask someone to explain why – 4 voters are needed to win a constituency and Magenta needs 4 constituencies, but there are less than 16 Magenta voters here.
Slides 19-21 5 minutes Slides 19-20 3 minutes	Tactical Voting	<i>Slides 22-23</i> Tactical voting can be used to avoid “wasted votes”, where votes did not make a difference in the outcome of the election. In the family pizza example, H and J votes for BBQ and anchovy, which nobody else voted for. <i>Click once</i> Instead, they may vote tactically for Four Cheese, rather than for their unpopular personal favourites, to avoid a ham and pineapple win. This is something that is common in real FPTP, where people vote for the most likely party to beat their least favourite, rather than their actual favourite candidate.
Slide 21 2 minutes (65)	Returning to Arrow’s Theorem	Ask students to discuss with their neighbours the extent to which they think First Past the Post is fair, considering Arrow’s theorem. You can then ask the students to vote, or share if they think First Past the Post voting is fair/not fair and why.
15 mins (80)	BREAK	Comfort break.
Slide 22 5 minutes (85)	What if we asked for more information?	Introduce students to the concept of a preferential or ranked-based voting system by showing them that the family members have now ranked all available pizzas, based on their first, second, third and fourth preferences. Ask students which topping would win here if they were using First Past the Post (Ham and Pineapple – with 4 of the 12 votes).

Slides & Time	Overview	Activity (see script for further details)
Slide 23 5 minutes (90)	Alternative vote (AV)	Introduce students to Alternative Vote (AV) – where the winner is found by reaching a majority after eliminated least popular candidates. Pictured is the Australian Parliament, which moved from being elected via FPTP to Alternative Vote in 1918. <i>More on this in the Additional Information document</i> If you want a quick demonstration, split the students in half and put them on Team Red or Team Blue. If there is an even number, you can move one student from one team to the other. If there is an odd number, keep one student in the middle and let them choose. The team that now has more people, has won, even though the difference is only one person.
Slides 24-27 5 minutes (95)	Alternative Vote (AV) example	<i>Slide 24</i> Show students the steps of how to calculate the winner in Alternative Vote, using the practice example of ice cream on the slides. Alternative Vote starts by calculating what the majority of votes is – ask students what the majority is (more than half the votes). Here, the majority would be 3. <i>Slide 25</i> No flavours have 3 votes. Strawberry and chocolate each have 2, so we eliminate vanilla, with the least votes. <i>Click to Slide 26 to show vanilla eliminated.</i> <i>Slide 27</i> After vanilla, strawberry is now C's first preference. Strawberry reaches the majority 3 votes and wins.
Slides 28 - 29 10 minutes (105)	Alternative vote (AV)/ 4 Cheese pizza wins! Worksheet	Hand out Worksheet 2: Alternative Vote (AV) Leave Slide 32 up to remind students of how AV works, until enough time for the Worksheet has passed. Ask students what pizza wins (cheese). <i>Slide 33</i> Ask students to discuss with neighbours how AV compares with FPTP or to put their hands up if they think it's a fairer/less fair system.
Slides 30-31 5 minutes (110)	Borda Count /Examples Worksheet 3	<i>Slide 30:</i> Explain how Borda works and how to calculate each ranking's points (this is on the slides.) Ask students if they can think of an example where Borda or a similar method is used, outside of elections. <i>Slide 31:</i> Show the examples on the slide – Formula 1 uses Borda to award points to racers and ultimately decide the winner, based on where they are placed in the race (25 for first, 18 for second etc.), and EuroVision similarly uses Borda to award points (12 for first, 10 for second, 8 for third etc.). Mention that Borda (and adaptations of it) is widely used within sports and competitions.

Slides & Time	Overview	Activity (see script for further details)
Slide 32-33 10 minutes (120)	Do we have a winner?	<i>Slide 32</i> Hand out Worksheet 3: Borda for students to work through <i>Slide 33</i> Ask student which pizza wins (there is a tie between Pepperoni and Cheese). Ask students how they would break the tie and if this makes Borda unfair. In an actual election, it is likely that a second election with only cheese and pepperoni would occur.
Slide 34-36 5 minutes (125)	Return to Arrow/ Arrow's Impossibility Theorem	<i>Slide 34</i> Quickly re-cap Arrow's theorem. Did any of these electoral systems satisfy all of Arrow's conditions? <i>Slide 35</i> Recap, one system at a time, the winner for each system, and an example of how that system has broken one of Arrow's rules. <i>Click to reveal each system in turn</i> <i>Slide 36</i> Tell students that Arrow's theorem is known as Arrow's impossibility theorem because he found that it is impossible for any electoral system to fulfil all criteria, when there are more than two candidates running.
Slide 37 5 minutes (130)	Can you come up with a better system?	Hand out Worksheet 4: Deciding on a system and the Ballot handout. You might need to adapt the following exercise depending on the number of students attending the masterclass. Divide students into groups of twelve students (this will serve as a new country) and, within these groups, divide students into groups of four. Go through instructions on slide and tell students they have 5 minutes to decide, in their groups of four, if they are going to create a system, adapt one of the ones discussed before (FPTP, AV or Borda) or pick a random one (in the next slide). You might want to have a visible timer going to keep them on track.
Slides 38-39 (130)	Random voting systems	The groups that want to pick a system randomly, pick a number between 1 and 9 and you reveal what voting system they will be testing, by switching slides. Hand out Random systems instructions to the group(s) who have chosen one.
Slide 40 10 minutes (140)	Mock election	Now that every group has a system in place, they will write instructions on the ballot handout for the members of the bigger/twelve-person group to vote and then distribute and collect these. Tell students they have ten minutes to do this, so they must be quick. You may wish to have a visible timer to keep them on track.

Slides & Time	Overview	Activity (see script for further details)
Slide 41 5 minutes (145)	Pick the country's system	Once all elections have happened, students will discuss in their groups of twelve and decide on one system to be used going forward.
Slide 42 5 minutes (150)	What system did you pick and why?	Each group of twelve students will feedback to the room which system they picked to run elections in their country and briefly explain why.
Slide 43 (last slide) 5 minutes (155)	Thank you! Feedback, tidy up, questions time Ask the Ri	Give a brief re-cap of the session. Don't forget to collect any questions which arise, and email them to us: masterclasses@ri.ac.uk We will send you answers as soon as possible. Then these can be reported back to the students at their next Masterclass session. In this way you cannot be "caught out" by a question. It also demonstrates the point that not everything in maths is known, but some questions need time and research to find answers sometimes, and sometimes the answer has not been found by anyone yet, of course! Maybe our Masterclass students will be the ones who solve the problem when they are older?